A Calculus of the One and the Many

Notes 01 • Version 01

I • Introduction

A. Perhaps no issue is as old, or has been wrestled with in so many parts of human inquiry (from science to theology) as that between the "one" and the "many."
B. Especially in mathematical frameworks, many intellectual (and most scientific) endeavours build in pre-supposed answers to or positions on the issue—e.g.
   1. Mathematical Δs[1] between sets and their members, functions and their values, etc.;
   2. Philosophical Δs between types, (written) tokens, (spoken) utterances, etc.;
C. Even in Christianity, where there is a tremendous amount of theological and intellectual discussion of the complex and even vexed issues of the "three" and the "one," the fundamental idea *that* there are three and one, and presuppositions about their respective identities, etc., can be dogmatic and rigid, rather than anything that is potentially dynamic, perspectival, inchoate, emergent, flexible, etc. …
D. Fundamentally, the aims of the fan calculus are:
   1. To bring to the fore, and to accord appropriate due, to the issue of what is one and what is many—i.e., of identity, individuality, singularity, multiplicity, and their relations.
   2. To allow answers or perspectives on the issue at least potentially to be fluid, perspectival, dynamic, and flexible in a variety of ways.
E. Fans
   1. One obvious challenge of developing a workable and "well-defined" calculus, which allows for flexible, dynamic, and potentially inchoate notions of identity, is not to confuse or conflate the *identity (or individuality) of the items in the calculus* with the *identity (or individuality)* of the stuff in the world that the elements of the calculus denote or represent.
   2. We will talk about that in depth, in due course.
   3. For now, however, I will simply state that we will use fan-shaped graphemes ⟦draw⟧ to indicate some stuff—a patch of reality—and to provide a way (with "points" or unities *in the calculus*) to represent the stuff as a unity, or to represent the stuff as a multiplicity (continuous or discrete).
   4. Examples
      a. Computer file: from some points of view "a file"; from others, Δ copies, etc.
      b. Web page: from some points of view, "one page", from another, Δ *languages*, Δ *copies downloaded to individual clients*, etc.
      c. Person: you *as an individual*, versus you *as a (4-D) space-time worm*.
F. Semantics
   1. Another empirical or substantive orientation affects (or infects) the calculus as I imagine it—which radically distinguishes it from, for example, set theory, model theory, and category theory.
   2. My belief is that object identity (individuality) arises out of acts of representation and abstraction (rather than being an intrinsic property of "stuff on its own."
   3. The primary use I will want to put the calculus to is to understand (and thus model)—at a very fine-grained level (since I think these things emerge and are stabilized very early) both (i) the details of language, reference, etc., and (ii) the structures of "abstraction," and how cognitive and other epistemic structures "abstract away from" the ultimately ineffable messy details of the, as it were, "bare world."
   4. So nods to or considerations of semantic and abstraction issues will permeate the design.

[1]Throughout, I use 'Δ' as short-hand for "differ", "distinction", "differing", etc.

G. Preparation
   1. Before talking about what the fan calculus is in detail—what it is designed for, what specific criteria it should meet, what phenomena in the world it should be held accountable to—I want to say something about calculi and mathematical systems in general. This is important because one of the aims is to be vastly more careful than is usual to discriminate "like" things, in order to unveil the fine-structure of our ontological assumptions and moves—moves that are often glossed over in more traditional frameworks.
   2. Here I will address four issues that come up with respect to the design of any calculus, on each of which we will want to agree on specific answers/approaches.

II • Four general points on Calculi

A. **Architecture and Generality**
   1. One issue of immense significance is the *generality* of a calculus
      a. To what extent it "builds in" assumptions that it (one) takes to hold of, or be appropriate to, *the entire domain under investigation*, or the "target empirical domain"—assumptions that are thereby assumed to hold of all objects/entities/fields/systems for which the calculus can be used as an explanatory device;
      b. To what extent it provides conceptual tools (also built into the architecture) for articulating or expressing variations between and among the different objects/entities/fields/systems it is used for; and
      c. To what extent it simply obliterates or renders invisible other dimensions, variations, etc.
   2. Examples
      a. "The calculus": ⟦ ◆ : undergraduate, physics; what is in "the calculus," what is in (e.g.) 2nd law of motion; allows comparison⟧[2]
      b. Linguistics: ⟦ ◆ : Δ theories that build full content built into architecture of calculus used to express them; as a result, no possibility to compare⟧
      c. Generality: ⟦ ◆ : Barwise, set theory … ⟧
   3. Bottom line: design issue of utmost seriousness. Division of labour …

B. **Semantics**
   1. Representation
      a. In a "mathematical" study of the calculus as an "object" of study, the calculus *is* the subject matter, about which Δs are drawn, claims made, etc.
      b. In *use*[3] however, a calculus is employed in order to represent, or express regularities within, an object (or subject) domain.
      c. If one uses language or graphics (as here) to notate the calculus, then there are two semantic relations of interest:
         i. Between the language and/or graphics and the corresponding mathematical entities or structures; and
         ii. Between those mathematical entities or structures and the task or object domain it is being used to, as it were, "model."
      d. I will use terminology as follows:

[2]'**⟦** ◆ … **⟧**' will be used to note places where stories are to be told …
[3]Generalizing ordinary usage, I will say that if I say "The watermelon is over there," I **use** the word 'watermelon' to **mention** an actual real-world watermelon. Whereas if I say "The word 'Boston' has either five or six letters, depending on how one individuates letters, I *use* the 8-character [sic] sequence ', B, o, s, t, o, n, and ' to *mention* the word 'Boston.' So "use" is a transitive verb whose (grammatical) objects is a word, used to refer to a context in which the word is employed by an intentionally-directed speaker or writer to refer to the object.

i. ‘**Model**’ to signify the relation between the mathematical structures (fans?) that we construct and the items in the object domains that we use the calculus to help us understand;
ii. ‘**Notate**’ to signify the relation between any linguistic and/or graphic items and the mathematical modeling structures
iii. ‘**Represent**’ more generally for a variety of semantic relations of interest …

e. Extreme care in noting and distinguishing semantic relations of these sorts is going to be hugely important to us. In fact as suggested §I.F, being able to track / express / make explicit the fine-structure of semantic relations is one of the aims of the calculus itself. So our discussions and use of it had better hew to very high standards …

2. Digressions
   a. Philosophy of science
      i. In ϕ[4] science, there has been a movement to identify theories with their *models*, rather than with any set of linguistic or symbolic expressions in anything language like.
      ii. This movement is called a “*semantic view*” because it takes the language to be about the model, and so recommends focus on the models as the right way to focus on what the theory is about, or anyway *what the theory is really saying* (and thus what Δs one theory from another).
      iii. It may be a great advance to focus on models instead of language, since there are so many “gratuitous” facts about languages that don’t bear on *what the theory says* about the world or domain it is a theory of.
      iv. But as the word ‘model’ betrays, a model of a phenomenon is still not the phenomenon itself ⟦ ◆ : Cf. picture from “The Limits of Correctness”—written 20 years ago.⟧
      v. My primary interest is in the relation between the model and the world—a relation that (as noted above) I *also* take to be a semantical one.
      vi. Hence I disagree with calling the popular contemporary model-focused view of theories of contemporary ϕ science “semantical.”
   b. Modelling: ⟦ ◆ : Talk about “model of the sentence” vs. “model of the world that the sentence is about”. Standard 3-way diagram; etc. ⟧
   c. Strictness: ⟦ ◆ : Semantical cleanliness at the price of usability, etc. ⇐ cf. 3-Lisp. Hugely important point.⟧

C. **Abstractness**

1. I talked above about “mathematical” entities and structures. ⟦Discussion?]
2. Many people think that mathematical constructions have purely abstract content (the numbers, pure sets, abstract mathematical functions, etc.).
3. On a strong version of such a view, one might think that a calculus[5] has *no empirical content*, and is therefore to be submitted only to a jury of consistency, or of integration with the rest of mathematics, etc.
4. This view might seem to be challenged by a view of (the late) Jon Barwise, who believed that every major scientific development—Newtonian mechanics, quantum mechanics, non-linear dynamics, etc.—had occasioned (brought about, necessitated) its own mathematical framework.
5. On the other hand, one can argue that Barwise’s view does not challenge the “no empirical content” claim, but instead say that the choice of *which* abstract (no empirical content) framework one is going to use, develop, exploit, etc., may be dictated or shaped by the particular empirical investi-

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[4] Throughout, ‘ϕ’ will be used for ‘philosophy,’ ‘philosophical,’ ‘philosopher,’ etc.
[5] I will use that work for any branch of mathematics the intent of which is to serve as a framework in terms of which to express fundamental laws of nature.

gation. That is, one might claim that empirical requirements at most mandate the choice of abstract framework.

6. But the abstractness has been rescued at the price of merely changing the terms of the discussion. Whereas an empirical mathematician might (i) claim that this or that feature (F) of a calculus or mathematical framework M was wrong, because it didn't correspond to some phenomenon P in the world, and therefore (ii) that M should be changed so as better to accommodate it (call the "fixed" version M'), an abstract mathematician might instead say (iii) that M wasn't wrong (assuming it wasn't inconsistent or didn't countervene what more generally we take to the case about mathematics), but instead assert (iv) that perhaps we merely preferred to use M', which was differently defined.
7. From a pragmatic point of view, however, these are more similar than Δ. I don't really care which one adopts. The point is that it is *demands of empirical adequacy*, not demands arising from the mathematical structures themselves, that drive us from M to M'. They are *empirical demands* reflecting requirements that are needed in order that we can do justice to the phenomenon for which the calculus is being used.
8. Bottom line: however abstract you take the fans and the fan calculus to be, our development / investigation of it will be empirically driven.
9. ⟦ ◆ : Example: set theory, Barwise's description:
   a. One motivating idea: something like a group or collection
   b. Two motivating themes
      i. Collection
      ii. Extension of a property or type: *those things that are α, for some α.*
      iii. That has led to Δ ⟧

D. **Aspect**

1. Intro
   a. It is important to keep in mind that only some of the infinite number of properties exemplified by any model M will correspond to only some of the infinite number of properties that hold of that which is modeled.
   b. Other properties of M may be *important* or *material*—e.g., having to do with the use of the model for epistemic purposes, and/or by the modeler.
   c. Overall, I will call the union of those properties that matter the **material** properties of M.
      i. ⟦ ◆ : Distinguish *material* and *physical* ⟧
   d. However the vast majority (measure 1?) of M's properties won't correspond in any way to anything that matters about D—i.e., won't correspond to any of the vast majority (measure 1?) properties of D, nor will they be of any epistemic or pragmatic use.
   e. Call these M's **immaterial** properties.
   f. Examples
      i. Balsa airplane models …
      ii. Set-theoretic models of the integers …
2. Modeling: A semantical account of the modeling relation that holds between M and D would be an account of the relation between the *material* properties of M and the modeled properties of D—or equivalently, the **material aspects of M** and the **modeled aspects of D**.
3. It is crucial not to take the immaterial properties of a model to be important as regards what is the case about D (or even worse, to *be* properties of D).
4. Discussion
   a. In prior work, I introduced a number of properties of models, having to do with the modeling relation (actually applicable to all representational relations).

b. Here I will introduce just one, which will very soon be applicable.
c. If a property P of M is used to model "the same property" (under some identity metric") of D, I will say that M **absorbs** P.
d. Examples
   i. Balsa airplane: *absorbs* geometry (under a scale factor)
   ii. "He woke up, made breakfast, bicycled to work, and died" (spoken)—absorbs temporal sequence.
   iii. Closed-world assumption in AI: absorbs object identity.

III • Classical ontology

A. Presumed—at least in much of ϕ, and arguably in everyday discourse, though I think that the latter is more contentious than most people (at last most ϕs realize)—that the world consists of *objects* that exemplify properties, stand in relations, are grouped together in sets, etc.
B. Call this **classical ontology**:
   1. *Objects* have their existence, and identity—in particular, their identity as one, as opposed to many—independent of, and explanatorily prior to
      a. The properties they exemplify
      b. The relations they stand in
      c. The sets they are grouped together in
      d. … etc.
   2. *Properties*—etymologically, things that are *proper* to the objects—are unitary in two senses:
      a. The property itself is a single thing (so that each of us exemplify the property of "being human"—where, crucially, it is the *same identical* property that we all exemplify.
      b. They hold *of* a single object.
   3. *Relations* are unitary in the first sense, and non-unitary in the second sense: they hold of two or more objects
   4. *Relational properties* are property-like things (unitary in both sense) which are defined in terms of a relation with one of its "slots" or "roles" filled in (e.g.: being 100 miles from Memphis).
   5. *Types*: like properties
C. In my experience: doesn't work all that well—for a whole variety of reasons:
   1. **Multiplicities:** things which, from one point of view seem singular, devolve, on closer inspection or from another point of view, into pluralities
      a. E.g., computer files: one file, various copies, even the copy itself multiplied, etc.
      b. Lots of Δ words have been developed for such cases:
         i. Copies, instances, editions, versions, releases, manifestations …
         ii. Types, tokens, instances, utterances, …
   2. **Non-conceptual:** things which are staples of everyday life but don't have stable or "well-defined" [reactionary!] identity conditions
      a. "The fog", which isn't exactly one, and not exactly a type, either
      b. "The mountains," which is somehow legitimately a plurality without the *identity* of the members of the plurality being well-established.
   3. **Ambiguities/equivocations:** Some things we call "objects," which seem to other things we call types or properties and refer only to their "instances,"
      a. Example: *asdf*
      b. Example: *words* and *names*
      c. Medium: *cars*

D. Discussion
   1. I want to bring these cases forward and do them appropriate justice.
   2. The guiding intuition is that they, and like cases, differ in virtue of involving Δs and nuances to be made within the calculus (not built into its architecture).
   3. That raises two design challenges:
      a. To build into the calculus' architecture what they have in common—that is: what *matters* about what they have in common; and
      b. To provide, still in the architecture, the mechanism to "articulate" (literally) their appropriate Δs and nuances.
   4. In addition, these "challenge" cases also involve uniformities that are shared by a number of cases that are included in classical ontology, but treated as distinct, such as (i) sets and their members, (ii) types and their instances, (iii) properties & relations and their extensions, etc.
   5. So the aim is to lift the distinctions between and among the classical types out of the architecture, so as to allow them to be expressed within it—which will
      a. Manifest what is common
      b. Require that what is distinctive be explicitly articulated
      c. Pave the way for variations more and less distinguished varieties.

IV • Examples

A. Intro
   1. Given all that preparation, I think the best way to proceed is simply to start drawing fans to represent a slew of different situations, and to see the various kinds of issues that come up.
   2. A couple of critical points, in advance
      a. As I take it will be obvious, the mathematical structures constitutive of the fan calculus are not meant to *absorb identity*.
      b. Nor are they intended to be *static* (the way in which the mathematical structures of differential equations used in physics are static—even though they represent dynamic situations).

B. Example [1] : Correspondence of names and objects
   1. ⟦ ◆ : Myth of "one name, one object"—except that names are types, objects (such as people) are not. ⟧
   2. ⟦ ◆ : "The skeleton of Mozart as a child." ⟧

C. Example [2] : Independence
   1. Traditionally, there are different views of what independence is, Δ by field:
      a. Probability: two events are independent just in case the probability of their joint occurrence is equal to the product of the probability of each separately
      b. Logic: two propositions are independent if the truth of one does not dictate or constrain the truth of the other.
      c. Algebra: two variables are independent if value of neither constrains value of other
      d. … etc.
   2. Fans will allow us a general characterization, from which the more particular ones follow, based on a "range of variation"—which is a form of "fan-out":
   3. ⟦ ◆ : draw ⟧

D. Example [3] : Synchronisation ⟦◆ : from paper⟧

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